Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 2. pp. 239 - 251
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.2.239

ON HARMONIC (h, r)-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce a new class of harmonic convex functions with respect to an arbitrary nonnegative function h and a parameter r, which is called harmonic (h, r)-convex functions. We establish some new Hermite-Hadamard type integral inequalities for harmonic (h, r)-convex functions. Some special cases are discussed, which appears to be new ones. Our results represent a significant refinement of the known cases. Ideas an techniques of this paper may motivate further research in this dynamic field.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 26D15,26A51

KEYWORDS AND PHRASES. r-convex functions, h-convex functions, *(h,r)*-convex function, Harmonic convex functions, Hermite-Hadamard type inequality.

1. INTRODUCTION

Convexity theory has become a rich source of inspiration in pure and applied sciences. This theory has not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general framework for studying a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [1, 6, 10, 12, 13, 14, 15, 16, 17, 25, 26, 27]

In recent years, convex functions have been generalized and extended in several directions using the novel and innovative techniques to study a wide class of unrelated problems in a unified framework. Varosanec [29], introduced the class of h-convex functions with respect to an arbitrary nonnegative function h. She has shown that this class contains some previously known classes of convex functions as special cases. Pearce et. al [24] investigated another class of convex functions, which is known as r-convex functions. This class of r-convex functions includes the convex functions and log-convex functions as special cases.

Definition 1.1. [24]. *Let r be a fixed real number. A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is r-convex, if, $\forall x, y \in I$ and $t \in [0, 1]$, we have*

$$f((1-t)a+tb) \leq \begin{cases} \{((1-t)[f(a)]^r + t[f(b)])^r\}^{\frac{1}{r}} & , r \neq 0 \\ (f(a))^{1-t}(f(b))^t & , r = 0. \end{cases}$$

It is clear that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

Ngoc et. al [12] obtained the Hermite-Hadamard inequality for r-convex functions.

Theorem 1.2. *Let $f \in HR(r, I)$ be r-convex function, where $a, b \in I$. If $f \in L[a, b]$. Then, for $r > 0$,*

$$\left[f\left(\frac{a+b}{2}\right)\right]^r \leq \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)^r \leq \left(\frac{r}{r+1}\right)^r([f(a)]^r + [f(b)]^r).$$

Hap and Vinh [10] established a Hermite-Hadamard inequality for *(h,r)*-convex functions.

Definition 1.3. [10]. *Let $r \neq 0$ be a real number and $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is (h,r)-convex function, or f belongs to the class $HR(h, r, I)$, if*

$$f\big((1-t)a + tb\big) \leq [h(1-t)[f(x)]^r + h(t)[f(y)]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in [0, 1].$$

Theorem 1.4. [10]. *Let $f \in HR(h, r, I)$, where $a, b \in I$. If $f \in L[a, b]$. Then, for $r > 0$,*

$$\frac{\left[f\left(\frac{a+b}{2}\right)\right]^r}{2h(\frac{1}{2})} \leq \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)^r \leq ([f(a)]^r + [f(b)]^r)\left(\int_0^1 [h(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r.$$

It is well known that harmonic means have important applications in various branches of pure and applied sciences such as circuit theory, game theory and geometric function theory. Harmonic means play significant role in the development of the parallel algorithms for solving nonlinear problems. It is known that harmonic convex functions are defined on the harmonic convex sets, which has emerged a significant and important generalization of the convex functions.. Several properties and characterizations of harmonic convex functions have been investigated and studied by Anderson et al. [1] and Iscan [11]. See also [19, 22] and the references therein.

Definition 1.5. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 1.6. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to harmonic convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

In particular, it has been shown [11] that f is a harmonic convex function, if and only if,

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a, b],$$

which is called Hermite-Hadamard inequality for harmonic convex function.

Noor and Noor[18, 19] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex sets can be characterized by a class of harmonic variational inequalities. This is a new problem and deserve further efforts to find the applications of the harmonic variational inequalities in various fields of pure and applied sciences.

Noor et al[20] introduced and studied a new class of harmonic r-convex functions. They have derived some interesting Hermite-Hadamard type inequalities for harmonic r-convex functions. We point out that the classes of h-convex functions, r-convex functions and harmonic r-convex functions are different generalizations of the convex functions. It is natural to study these classes of convex functions in a unified manner. Inspired and motivated by the ongoing research, we introduce the concept of harmonic (h, r)-convex function with respect to an arbitrary nonnegative function h. This class is more general and contains several new classes of harmonic r-convex functions such as Breckner type of s-harmonic r-convex functions, Godunova-Levin type of s-harmonic r-convex functions and harmonic P-r functions. We discuss some properties of harmonic (h, r)-convex function. We establish several Hermite-Hadamard inequalities for harmonic (h, r)-convex function. One can derive several Hermite-Hadamard type inequalities for our main results for different classes of harmonic convex functions. The readers are encouraged to find the applications of harmonic $(h, r)-$ convex functions in various fields of pure and applied sciences.

We now introduce some new concepts.

Definition 1.7. *Let r be a real number and $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is harmonic (h,r)-convex function, or f belongs to the class $HR(h, r, I)$, if. $\forall x, y \in I$,*

$$(1)\quad f\left(\frac{xy}{tx+(1-t)y}\right) \leq \begin{cases} \{(h(1-t)[f(a)]^r + h(t)[f(b)])^r\}^{\frac{1}{r}} & ,r \neq 0 \\ (f(a))^{1-t}(f(b))^t & ,r = 0. \end{cases}$$

The function f is said to be harmonic (h, r)-concave function, if and only if, -f is harmonic (h, r)-convex function.

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right)^r \leq \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}\left([f(x)]^r + [f(y)]^r\right)^{\frac{1}{r}}, \quad \forall x, y \in I,$$

which is called Jensen harmonic (h, r)-convex function.

Now we discuss some special cases of harmonic (h, r) convex function, which appears to be new ones.

I. If we take $h(t) = t$ in Definition 1.7, then it reduces to the Definition of harmonic r-convex functions, which was introduced by Noor et al[20].

Definition 1.8. *Let $r \neq 0$ be a real number. We say that $f : I = [a, b] \subseteq \mathbb{R}\setminus \{0\} \to \mathbb{R}$ is harmonic r-convex function, or f belongs to the class $HR(r, I)$, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq [(1-t)[f(x)]^r + t[f(y)]^r]^{\frac{1}{r}}, \qquad \forall x, y \in I, t \in [0, 1].$$

If $r = 1$ in Definition 1.7, then it reduces to the Definition of harmonic h-convex functions, see [11].

II. If $h(t) = t^s$ in Definition 1.7 then it reduces to the Definition of Breckner type of s-harmonic r-convex functions.

Definition 1.9. *Let $r \neq 0$ be a real number. We say that $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is harmonic s-r-convex function, or f belongs to the class $HR(s,r,I)$ and $s \in (0,1)$, we have*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq [(1-t)^s[f(x)]^r + t^s[f(y)]^r]^{\frac{1}{r}}, \qquad \forall x,y \in I, t \in [0,1].$$

III. If $h(t) = t^{-s}$ in Definition 1.7, then it reduces to the Definition of Godunova-Levin type of s-harmonic r-convex functions.

Definition 1.10. *Let $r \neq 0$ be a real number. A function $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is Godunova-Levin type of s-harmonic r-convex functions, or f belongs to the class $HR(-s,r,I)$, and $s \in (0,1)$, we have*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq [(1-t)^{-s}[f(x)]^r + t^{-s}[f(y)]^r]^{\frac{1}{r}}, \qquad \forall x,y \in I, t \in (0,1).$$

It is obvious that for $s = 0$, s-harmonic Godunova-Levin r-convex functions reduce to harmonic P-r-convex functions. If $s = 1$, then s-harmonic Godunova-Levin r-convex functions reduce to harmonic Godunova-Levin r-convex functions.

Definition 1.11. [25]. *Two functions f, g are said to be similarly ordered (f is g-monotone), if and only if,*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0, \qquad \forall x,y \in \mathbb{R}^n.$$

One can easily show that if f and g are two similarly ordered harmonic (h,r)-convex functions and $h(1-t)+h(t) \leq 1$, then the product fg is again a harmonic (h,r)-convex function, see, for example, Noor et al[20].

The Euler Beta function is a special function defined as:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \forall x,y > 0.$$

2. Main results

In this section, we obtain Hermite-Hadamard inequality for harmonic (h,r)-convex function. Let I and J are intervals in $\mathbb{R}$, $[0,1] \subseteq J$, functions h, k are real positive defined on J and f, g are real positive functions defined on I.
We recall the following well known fact [6], which plays a crucial part in obtaining the main results.

Fact[6]**:** If the the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ is defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is harmonic *(h,r)*-convex on $[a,b]$, if and only if, g is *(h,r)*-convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Theorem 2.1. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic (h,r)-convex function. If $f \in L[a, b]$ then*

$$\frac{1}{2h(\frac{1}{2})}\left[f\left(\frac{2ab}{a+b}\right)\right]^r \le \left(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right)^r \le ([f(a)]^r + [f(b)]^r)\left(\int_0^1 h(t)^{\frac{1}{r}}\mathrm{d}t\right)^r.$$

Proof. We consider Hermite-Hadamard inequality for (h, r)-convex function $g(t) = f\left(\frac{1}{t}\right)$ on the closed interval $[\frac{1}{b}, \frac{1}{a}]$, that is,

$$\frac{1}{2h(\frac{1}{2})}\left[f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}}\right)\right]^r \le \left(\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)\mathrm{d}t\right)^r \le ([f(a)]^r + [f(b)]^r)\left(\int_0^1 h(t)^{\frac{1}{r}}\mathrm{d}t\right)^r.$$

Using change of variable $x = \frac{1}{t}$, we obtain the Hermite-Hadamard inequality for harmonic (h, r)-convex functions. □

We now discuss some special cases of Theorem 2.1 which appears to new ones.

I. If $h(t) = t$, then Theorem 2.1 reduces to the following result.

Corollary 2.2. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic r-convex function. If $f \in L[a, b]$, then*

$$\left[f\left(\frac{2ab}{a+b}\right)\right]^r \le \left(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right)^r \le ([f(a)]^r + [f(b)]^r)\left(\frac{r}{r+1}\right)^r.$$

II. If $h(t) = t^s$, then Theorem 2.1 reduces to the following result.

Corollary 2.3. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic s-r-convex function. If $f \in L[a, b]$ and $s \in (0, 1)$, then*

$$2^{s-1}\left[f\left(\frac{2ab}{a+b}\right)\right]^r \le \left(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right)^r \le ([f(a)]^r + [f(b)]^r)\left(\frac{r}{r+s}\right)^r.$$

III. If $h(t) = t^{-s}$, then Theorem 2.1 reduces to the following result.

Corollary 2.4. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic s-r Godunova-Levin function. If $f \in L[a, b]$ and $s \in (0, 1)$, then*

$$\frac{1}{2^{s+1}}\left[f\left(\frac{2ab}{a+b}\right)\right]^r \le \left(\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right)^r \le ([f(a)]^r + [f(b)]^r)\left(\frac{r}{r-s}\right)^r.$$

Theorem 2.5. *Let $f \in (h, r, I)$ and $g \in (k, r, I)$ with $a < b$. Then, for $r > 0$,*

$$\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r \le M(a,b)\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + N(a,b)\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r,$$

where

$$M(a,b) = [f(a)g(a)]^r + [f(b)g(b)]^r, \tag{2}$$

$$(3) \qquad N(a,b) = [f(a)g(b)]^r + [f(b)g(a)]^r.$$

Proof. Since $f \in (h,r,I)$ and $g \in (k,r,I)$ with $a<b$, where $r>0$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right) \le [h(1-t)[f(a)]^r + h(t)[f(b)]^r]^{\frac{1}{r}}$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \le [k(1-t)[g(a)]^r + k(t)[g(b)]^r]^{\frac{1}{r}}$$

Consider

$$\begin{aligned} & f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ \le\ & [h(1-t)k(1-t)[f(a)g(a)]^r + h(1-t)k(t)[f(a)g(b)]^r + h(t)k(1-t)[f(b)g(a)]^r \\ & + h(t)k(t)[f(b)g(b)]^r]^{\frac{1}{r}} \end{aligned}$$

Using Minkowskis inequality, we have

$$\begin{aligned} & \left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r = \left(\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\right)^r \\ \le\ & (\int_0^1 [h(1-t)k(1-t)[f(a)g(a)]^r + h(t)k(1-t)[f(b)g(a)]^r + h(1-t)k(t)[f(a)g(b)]^r \\ & + h(t)k(t)[f(b)g(b)]^r]^{\frac{1}{r}}\mathrm{d}t)^r \\ \le\ & [f(a)g(a)]^r\left(\int_0^1 [h(1-t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + [f(a)g(b)]^r\left(\int_0^1 [h(1-t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\ & + [f(b)g(a)]^r\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + [f(b)g(b)]^r\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\ =\ & \Big([f(a)g(a)]^r + [f(a)g(a)]^r\Big)\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\ & + \Big([f(a)g(b)]^r + [f(b)g(a)]^r\Big)\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \\ =\ & M(a,b)\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + N(a,b)\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r, \end{aligned}$$

which is the required result. □

We now some special cases,

I. If we take $h(t) = k(t) = t$, then Theorem 2.5 reduces to the following new result.

Corollary 2.6. *Let $f, g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic r-convex functions on $[a,b]$ with $a<b$ and $r>0$. Then*

$$\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r \le M(a,b)\left(\frac{r}{r+2}\right)^r + N(a,b)\left(B\left(\frac{1}{r}+1,\frac{1}{r}+1\right)\right)^r.$$

II. If we take $h(t) = k(t) = t^s$, then Theorem 2.5 reduces to the following new result.

Corollary 2.7. *Let $f, g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic s-r-convex functions on $[a, b]$ with $a < b$ and $r > 0$. Then*

$$\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r \le M(a,b)\left(\frac{r}{r+2s}\right)^r + N(a,b)\left(B\left(\frac{s}{r}+1, \frac{s}{r}+1\right)\right)^r.$$

III. If we take $h(t) = k(t) = t^{-s}$, then Theorem 2.5 reduces to the following new result.

Corollary 2.8. *Let $f, g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic s-r Godunova-Levin functions on $[a, b]$ with $a < b$ and $r > 0$. Then*

$$\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r \le M(a,b)\left(\frac{r}{r+2s}\right)^r + N(a,b)\left(B\left(\frac{-s}{r}+1, \frac{-s}{r}+1\right)\right)^r.$$

Theorem 2.9. *Let $f \in (h, r_1, I)$ and $g \in (k, r_2, I)$ with $a < b$. Then, for $r_1 > 0$, and $r_2 > 0$, we have*

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \quad \le \quad & \frac{1}{2}\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}}\int_0^1 [h(t)]^{\frac{2}{r_1}}\mathrm{d}t \\ & + \frac{1}{2}\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}\int_0^1 [k(t)]^{\frac{2}{r_2}}\mathrm{d}t.\end{aligned}$$

Proof. Since $f \in (h, r, I)$ and $g \in (k, r, I)$ with $(r_1 > 0,\ r_2 > 0)$, we have

$$\begin{aligned} f\left(\frac{ab}{ta+(1-t)b}\right) &\le [h(1-t)[f(a)]^{r_1} + h(t)[f(b)]^{r_1}]^{\frac{1}{r_1}} \\ g\left(\frac{ab}{ta+(1-t)b}\right) &\le [k(1-t)[g(a)]^{r_2} + k(t)[g(b)]^{r_2}]^{\frac{1}{r_2}} \end{aligned}$$

for all $t \in [0, 1]$. Now multiplying the above inequalities and integrating over $[0, 1]$, we have

$$\begin{aligned} &\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x = \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\ \le\ & \int_0^1 [h(1-t)[f(a)]^{r_1} + h(t)[f(b)]^{r_1}]^{\frac{1}{r_1}}[k(1-t)[g(a)]^{r_2} + k(t)[g(b)]^{r_2}]^{\frac{1}{r_2}}\mathrm{d}t. \end{aligned}$$

Using Cauchys inequality, we have

$$\begin{aligned} & \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\ \le\ & \frac{1}{2}\int_0^1 [h(1-t)[f(a)]^{r_1} + h(t)[f(b)]^{r_1}]^{\frac{2}{r_1}}\mathrm{d}t \\ +\ & \frac{1}{2}\int_0^1 [k(1-t)[g(a)]^{r_2} + k(t)[g(b)]^{r_2}]^{\frac{2}{r_2}}\mathrm{d}t. \end{aligned}$$

Using Minkowskis inequality, we have

$$\begin{aligned}
&\int_0^1 [h(1-t)[f(a)]^{r_1} + h(t)[f(b)]^{r_1}]^{\frac{2}{r_1}}\mathrm{d}t \\
\leq\ & \left[\left(\int_0^1 h(1-t)^{\frac{2}{r_1}}[f(a)]^2\mathrm{d}t\right)^{\frac{r_1}{2}} + \left(\int_0^1 h(t)^{\frac{2}{r_1}}[f(b)]^2\mathrm{d}t\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
=\ & \left[[f(a)]^{r_1}\left(\int_0^1 h(1-t)^{\frac{2}{r_1}}\mathrm{d}t\right)^{\frac{r_1}{2}} + [f(b)]^{r_1}\left(\int_0^1 h(t)^{\frac{2}{r_1}}\mathrm{d}t\right)^{\frac{r_1}{2}}\right]^{\frac{2}{r_1}} \\
=\ & \left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \int_0^1 [h(t)]^{\frac{2}{r_1}}\mathrm{d}t.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 [k(1-t)[g(a)]^{r_2} + k(t)[g(b)]^{r_2}]^{\frac{2}{r_2}}\mathrm{d}t \\
\leq\ & \left[\left(\int_0^1 (k(1-t)^{\frac{2}{r_2}}[g(a)]^2\mathrm{d}t\right)^{\frac{r_2}{2}} + \left(\int_0^1 k(t)^{\frac{2}{r_2}}[g(b)]^2\mathrm{d}t\right)^{\frac{r_2}{2}}\right]^{\frac{2}{r_2}} \\
=\ & \left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}} \int_0^1 [k(t)]^{\frac{2}{r_2}}\mathrm{d}t.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \leq\ & \frac{1}{2}\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \int_0^1 [h(t)]^{\frac{2}{r_1}}\mathrm{d}t \\
&+\frac{1}{2}\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}} \int_0^1 [k(t)]^{\frac{2}{r_2}}\mathrm{d}t,
\end{aligned}$$

which is the required result. □

Special Cases:

I. If we take $h(t) = k(t) = t$, then, Theorem 2.9 reduces to the following new result.

Corollary 2.10. *Let $f, g : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic r_1-convex and harmonic r_2-convex functions respectively on $[a, b]$ with $a < b$. Then $r_1 > 0$ and $r_2 > 0$, we have*

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \leq\ & \frac{1}{2}\left(\frac{r_1}{r_1+2}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\
&+\frac{1}{2}\left(\frac{r_2}{r_2+2}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.
\end{aligned}$$

II. In Theorem 2.9, if $r_1 = r_2 = 2$, we have

Corollary 2.11. *Let $f, g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic 2-convex functions on $[a, b]$ with $a < b$. Then,*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \leq \frac{1}{4}\Big[[f(a)]^2 + [f(b)]^2 + [g(a)]^2 + [g(b)]^2\Big].$$

III. In Theorem 2.9, if $r_1 = r_2 = 2$ and $f(x) = g(x)$, then, we have

Corollary 2.12. *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic 2-convex function on $[a, b]$ with $a < b$. Then,*

$$\frac{ab}{b-a}\int_a^b \frac{[f(x)]^2}{x^2}\mathrm{d}t \leq \frac{1}{2}\bigg[[f(a)]^2 + [f(b)]^2\bigg].$$

IV. If we take $h(t) = k(t) = t^s$, then Theorem 2.9 reduces to the following new result.

Corollary 2.13. *Let $f, g : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic r_1-s-convex and harmonic r_2-s-convex functions respectively on $[a, b]$ with $a < b$. Then for $r_1 > 0$ and $r_2 > 0$, we have*

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \leq\ & \frac{1}{2}\left(\frac{r_1}{r_1+2s}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\ & + \frac{1}{2}\left(\frac{r_2}{r_2+2s}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.\end{aligned}$$

V. If we take $h(t) = k(t) = t^{-s}$, then Theorem 2.9 reduces to the following new result.

Corollary 2.14. *Let $f, g : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic r_1-s Godunova-Levin and harmonic r_2-s Godunova-Levin functions respectively on $[a, b]$ with $a < b$. Then for $r_1 > 0$ and $r_2 > 0$, we have*

$$\begin{aligned}& \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\ \leq\ & \frac{1}{2}\left(\frac{r_1}{r_1-2s}\right)\left[[f(a)]^{r_1} + [f(b)]^{r_1}\right]^{\frac{2}{r_1}} \\ & + \frac{1}{2}\left(\frac{r_2}{r_2-2s}\right)\left[[g(a)]^{r_2} + [g(b)]^{r_2}\right]^{\frac{2}{r_2}}.\end{aligned}$$

Theorem 2.15. *Let $f \in (h, r_1, I)$ and $g \in (k, r_2, I)$ with $a < b$. Then, for $r_1 > 0$, $r_2 > 0$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$, we have*

$$\begin{aligned}& \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\ \leq\ & \Big([f(a)]^{r_1} + [f(b)]^{r_1}\Big)^{\frac{1}{r_1}}\Big([g(a)]^{r_2} + [g(b)]^{r_2}\Big)^{\frac{1}{r_2}}\left(\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{r_1}} \\ & \left(\int_0^1 k(t)\mathrm{d}t\right)^{\frac{1}{r_2}}.\end{aligned}$$

Proof. Since $f \in (h, r_1, I)$ and $g \in (k, r_2, I)$ with ($r_1 > 0$, $r_2 > 0$), we have

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq [h(1-t)[f(a)]^{r_1} + h(t)[f(b)]^{r_1}]^{\frac{1}{r_1}}$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq [k(1-t)[g(a)]^{r_2} + k(t)[g(b)]^{r_2}]^{\frac{1}{r_2}}$$

for all $t \in [0,1]$. Now multiplying the above inequalities and integrating over $[0,1]$, we have

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x = \int_0^1 f\Big(\frac{ab}{ta+(1-t)b}\Big)g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t$$
$$\leq \int_0^1 [h(1-t)[f(a)]^{r_1}+h(t)[f(b)]^{r_1}]^{\frac{1}{r_1}}[k(1-t)[g(a)]^{r_2}+k(t)[g(b)]^{r_2}]^{\frac{1}{r_2}}\mathrm{d}t.$$

Using Holder's inequality, we have

$$\int_0^1 [h(1-t)[f(a)]^{r_1}+h(t)[f(b)]^{r_1}]^{\frac{1}{r_1}}[k(1-t)[g(a)]^{r_2}+k(t)[g(b)]^{r_2}]^{\frac{1}{r_2}}\mathrm{d}t$$
$$\leq \Big(\int_0^1 [h(1-t)[f(a)]^{r_1}+h(t)[f(b)]^{r_1}]\mathrm{d}t\Big)^{\frac{1}{r_1}}\Big(\int_0^1 [k(1-t)[g(a)]^{r_2}+k(t)[g(b)]^{r_2}]\mathrm{d}t\Big)^{\frac{1}{r_2}}$$
$$= \Big([f(a)]^{r_1}\int_0^1 h(1-t)\mathrm{d}t+[f(b)]^{r_1}\int_0^1 h(t)\mathrm{d}t\Big)^{\frac{1}{r_1}}$$
$$\Big([g(a)]^{r_2}\int_0^1 k(1-t)\mathrm{d}t+[g(b)]^{r_2}\int_0^1 k(t)\mathrm{d}t\Big)^{\frac{1}{r_2}}$$
$$= \Big([f(a)]^{r_1}+[f(b)]^{r_1}\Big)^{\frac{1}{r_1}}\Big([g(a)]^{r_2}+[g(b)]^{r_2}\Big)^{\frac{1}{r_2}}\Big(\int_0^1 h(t)\mathrm{d}t\Big)^{\frac{1}{r_1}}\Big(\int_0^1 k(t)\mathrm{d}t\Big)^{\frac{1}{r_2}}.$$

Thus

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x$$
$$\leq \Big([f(a)]^{r_1}+[f(b)]^{r_1}\Big)^{\frac{1}{r_1}}\Big([g(a)]^{r_2}+[g(b)]^{r_2}\Big)^{\frac{1}{r_2}}\Big(\int_0^1 h(t)\mathrm{d}t\Big)^{\frac{1}{r_1}}$$
$$\Big(\int_0^1 k(t)\mathrm{d}t\Big)^{\frac{1}{r_2}},$$

which is the required result. □

For suitable and appropriate choice of r_1, r_2 and the function h, one can obtain several new integral inequalities for various classes of harmonic (h,r)-convex functions.

Theorem 2.16. *Let $f \in (h,r,I)$ and $g \in (k,r,I)$ with $a < b$. Then, for $r > 0$, we have*

$$\Big[f\Big(\frac{2ab}{a+b}\Big)g\Big(\frac{2ab}{a+b}\Big)\Big]^r - 2\Big[h\Big(\frac{1}{2}\Big)k\Big(\frac{1}{2}\Big)\Big]\Big(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\Big)^r$$
$$\leq 2\Big[h\Big(\frac{1}{2}\Big)k\Big(\frac{1}{2}\Big)\Big]\Big[M(a,b)\Big(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\Big)^r$$
$$+N(a,b)\Big(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\Big)^r\Big].$$

where $M(a,b)$ and $N(a,b)$ are given by (2) and (3) respectively.

Proof. Let $f \in (h,r,I)$ and $g \in (k,r,I)$ with $t=\frac{1}{2}$. Then, we have

$$f\left(\frac{2xy}{x+y}\right) \le \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left[[f(x)]^r+[f(y)]^r\right]^{\frac{1}{r}}, \qquad \forall x,y\in I.$$

$$g\left(\frac{2xy}{x+y}\right) \le \left[k\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left[[g(x)]^r+[g(y)]^r\right]^{\frac{1}{r}}, \qquad \forall x,y\in I.$$

Let $x=\frac{ab}{ta+(1-t)b}$, and $y=\frac{ab}{(1-t)a+tb}$. Then

$$f\left(\frac{2ab}{a+b}\right) \le \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left(\left[f\left(\frac{ab}{ta+(1-t)b}\right)\right]^r + \left[f\left(\frac{ab}{(1-t)a+tb}\right)\right]^r\right)^{\frac{1}{r}}.$$

$$g\left(\frac{2ab}{a+b}\right) \le \left[k\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left(\left[g\left(\frac{ab}{ta+(1-t)b}\right)\right]^r + \left[g\left(\frac{ab}{(1-t)a+tb}\right)\right]^r\right)^{\frac{1}{r}}.$$

Put $\bar{a}=\frac{ab}{ta+(1-t)b}$ and $\bar{b}=\frac{ab}{(1-t)a+tb}$, we have

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\
\le\ & \left[h\left(\frac{1}{2}\right)k\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\right)^{\frac{1}{r}} \\
=\ & \left[h\left(\frac{1}{2}\right)k\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \int_0^1 \left([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\right)^{\frac{1}{r}} \mathrm{d}t.
\end{aligned}$$

Using Minkowskis inequality, we have

$$\begin{aligned}
& \left(\int_0^1 \left([f(\bar{a})]^r[g(\bar{a})]^r+[f(\bar{b})]^r[g(\bar{b})]^r+[f(\bar{a})]^r[g(\bar{b})]^r+[f(\bar{b})]^r[g(\bar{a})]^r\right)^{\frac{1}{r}}\right)^r \mathrm{d}t \\
\le\ & \left(\int_0^1 [f(\bar{a})][g(\bar{a})]\mathrm{d}t\right)^r + \left(\int_0^1 [f(\bar{b})][g(\bar{b})]\mathrm{d}t\right)^r \\
& + \left(\int_0^1 [f(\bar{a})][g(\bar{b})]\mathrm{d}t\right)^r + \left(\int_0^1 [f(\bar{b})][g(\bar{a})]\mathrm{d}t\right)^r \\
\le\ & \left(\int_0^1 [f(\bar{a})][g(\bar{a})]\mathrm{d}t\right)^r + \left(\int_0^1 [f(\bar{a})][g(\bar{b})]\mathrm{d}t\right)^r \\
& + \left(\int_0^1 (h(1-t)[f(a)]^r+h(t)[f(b)]^r)^{\frac{1}{r}}(k(t)[g(a)]^r+k(1-t)[g(b)]^r)^{\frac{1}{r}}\mathrm{d}t\right)^r \\
& + \left(\int_0^1 (h(t)[f(a)]^r+h(1-t)[f(b)]^r)^{\frac{1}{r}}(k(1-t)[g(a)]^r+k(t)[g(b)]^r)^{\frac{1}{r}}\mathrm{d}t\right)^r
\end{aligned}$$

$$\begin{aligned}
\le\ & 2\left[\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r + M(a,b)\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r \right. \\
& \left. + N(a,b)\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r\right]
\end{aligned}$$

This implies

$$\left[f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\right]^r - 2\left[h\left(\frac{1}{2}\right)k\left(\frac{1}{2}\right)\right]\left(\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\right)^r$$
$$\leq 2\left[h\left(\frac{1}{2}\right)k\left(\frac{1}{2}\right)\right]\left[M(a,b)\left(\int_0^1 [h(t)k(1-t)]^{\frac{1}{r}}\mathrm{d}t\right)^r + N(a,b)\left(\int_0^1 [h(t)k(t)]^{\frac{1}{r}}\mathrm{d}t\right)^r\right],$$

which is the required result. □

Remark. For suitable and appropriate choice of r_1, r_2 and the function h, one can obtain several new integral inequalities for various classes of harmonic (h, r)-convex functions. We leave these details to the interested readers.

Acknowledgements

The authors would like to thank Dr. S. M. Junaid Zaidi (H.I., S.I.), Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environments. This research is supported by HEC project Grant: NRPU-No. 20-1966/R&D/11-2253.

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